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THE COMPLETE INTERSECTION LOCUS OF CERTAIN IDEALS

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Introduction

Let R be a Noetherian local ring with maximal ideal m and dimension d. The most primitive quantities attached to an ideal I are its height, $\mathrm{ht}(I)$, and its minimum number of generators, $\upsilon(I)$. The fundamental relation between $\mathrm{ht}(I)$ and $\upsilon(I)$ is provided by Krull's theorem:

(1) $\qquad \upsilon(I) \geq \mathrm{ht}(I).$

The case of equality is noteworthy enough and I will be called a complete intersection if it so happens. (Strictly, this terminology is used when the ideal is generated by a regular sequence, but the two notions coincide in what will be our standard context – that of Cohen–Macaulay rings.)

To bridge the gap between $\upsilon(I)$ and $\mathrm{ht}(I)$ several other measures have been introduced. We single out:

(i) $\mathrm{ara}(I)$ = arithmetical rank of I := the least integer r such that the radical of I is the radical of an r-generated ideal.

(ii) $l(I)$ = analytic spread of I := Krull dimension of the algebra $\bigoplus I^t \otimes R/m$, that is, $1+$ degree of the (Hilbert) polynomial that reads $\dim_{R/m}(I^t/mI^t)$, $t \gg 0$. If the residue field of R is infinite – an innocous hypothesis – $l(I)$ is the number of generators of the smallest ideal J such that $I^{t+1} = J \cdot I^t$ for some t.

(iii) $\mathrm{cd}(I)$ = cohomological dimension of $I := \sup\{j \mid H_I^j(R) \neq 0\}$.

The inequality above refines then to [19]:

(2) $\qquad \upsilon(I) \geq l(I) \geq \mathrm{ara}(I) \geq \mathrm{cd}(I) \geq \mathrm{ht}(I).$

Another natural approach to the comparison between $\mathrm{ht}(I)$ and $\upsilon(I)$ – and for simplicity we now assume I to be height unmixed – is through the properties of the

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set of primes P where the localization I_P is a complete intersection. This defines an open set $D(J)$ of Spec(R) and will be denoted CIL(I). If R is a Cohen–Macaulay ring and P is a minimal prime of J, then I/I^2 determines a vector bundle on the punctured spectrum of the local ring $(R/I)_P$. The codimension of CIL(I), ht(J), shall be denoted by $c(I)$. One of our aims here is to examine how some of the numbers above are reflected on $c(I)$.

Broadly speaking our aim here is to estimate $c(I)$ for ideals where the conormal module, I/I^2, is 'sufficiently well known'. An extreme case of this formulation is the following conjecture: Let I be an ideal of the regular local ring R. If I/I^2 has finite projective dimension over R/I, then I is a complete intersection. From the cases settled thus far the emerging evidence for it and for the similar conjectures on the other (co-)normal modules is quite strong. Furthermore we shall be interested in those fragments of the hypothesis above that lead to sharp estimates of $c(I)$ in more general cases.

We shall now describe the contents of this paper. In Section 1 we recall a general estimate of Faltings [8] on $c(I)$, as recently improved by Huneke [15]. We shall also derive a bound for $c(I)$ in terms of the number of generators and relations of I. These results shall provide the backdrop for the especial estimates of the paper proper.

In Section 2 we discuss an enrichment by Gulliksen [10] of the Tate's resolution of R/I. A minor rephrasing of his proof leads, essentially, to the following assertion: Let R be a local ring and let I be an ideal of finite projective dimension; the first homology module, H_1, of the Koszul complex K built on a minimal generating set of generators of I does not admit a nonzero (R/I)-free summand. As a consequence one can write bounds for $c(I)$ that may, under certain conditions, be much sharper than those of Section 1.

The conjectural homological rigidity for I/I^2 (cf. [24], [25]) is stated in Section 3 – i.e. whether for an ideal I of a regular ring R, 0 and ∞ are the only possible values for $\mathrm{pd}_{R/I}(I/I^2)$. Along with analogs for the other (co-)normal modules it asks whether local complete intersections are characterized by the finiteness of the projective dimension of these modules. The results of Section 2 could then be stated as asserting that $\mathrm{pd}_{R/I}(I/I^2) \neq 1$.

In the next section it is shown that the canonical module of R/I, for the ideals in the conjecture, has the expected form: it is cyclic. This says that the Cohen–Macaulay ideals in the conjecture are in fact Gorenstein ideals, and settles the question in several cases – e.g. arbitrary ideals of height two in rings containing a field.

In Section 5 we use available information on the depths of the Koszul homology modules to resolve other instances of the conjecture. Finally we discuss the form the estimates of Section 1 assume when applied to ideals satisfying $\mathrm{pd}_{R/I}(I/I^2) \leq 3$. It allows for extending the catalogue of solved cases up to various Cohen–Macaulay ideals of height four.

1. General bounds

There are some surprisingly sharp bounds on $c(I)$ that result directly from the intersection theorem of Serre and of novel forms of the theorem of Krull. We begin with the latter.

The following result of Bruns [3] represents the breaking down of the classical estimates of Eagon–Northcott [5] for the size of determinantal ideals.

(1.1) **Theorem.** *Let R be a commutative Noetherian ring and let ϕ be an $m\times n$ matrix with entries in R. For each integer r, denote by $I_r(\phi)$ the ideal generated by the r-sized minors of ϕ. If $I_t(\phi)\neq R$ and $I_{t+1}(\phi)=0$, then* $\operatorname{ht}(I_t(\phi))\leq m+n-2t+1$.

This has the following consequence on $c(I)$.

(1.2) **Corollary.** *Let R be a Cohen–Macaulay ring and let I be an unmixed ideal that is not a local complete intersection. If I admits a presentation*

$$R^m \xrightarrow{\Phi} R^n \to I \to 0.$$

Then $c(I)\leq m-n+3\operatorname{ht}(I)+1$.

Proof. Tensoring the presentation of I by R/I and applying (1.1) to $\phi=\Phi\otimes R/I$ we get (with $g=\operatorname{ht}(I)$).

$$\operatorname{ht}(I_{n-g}(\phi))\leq m+n-2(n-g)+1.$$

Since $I_{n-g}(\phi)$ defines the free locus of I/I^2, reading this estimate in $\operatorname{Spec}(R)$ we get the asserted inequality. □

Remarks. (a) If I is a perfect ideal of height 2, then $m=n-1$, so that $c(I)\leq 6$ (cf. [22]).

(b) If I is a Gorenstein ideal of height 3, then one obtains $c(I)\leq 10$. This is the estimate of [4] and it follows from their structure theorem that this bound is sharp. If I is just assumed to be Cohen–Macaulay, from [11] one can conclude that there is enough room – i.e. with $c(I)\leq 10$ – to make I Gorenstein.

When the number of relations, m, is large, this estimate is rather poor. For small $n=\upsilon(I)$ the following result of Faltings [8] – as recently improved upon by Huneke [15] – furnishes very good bounds.

(1.3) **Theorem.** *Let R be a regular local ring and let I be an unmixed ideal. If I is not a complete intersection, then* $c(I)\leq l(I)+2\operatorname{ht}(I)-1$.

Remark. If $\upsilon(I)=g+1$ ($g=\operatorname{ht}(I)$) – i.e. if I is an almost complete intersection, then $c(I)\leq 3g$.

If I is Cohen–Macaulay this estimate may be improved (for $g>2$) down to $c(I)\leq 3+2g$. Indeed for this bound it follows from [11] that I must be a Gorenstein ideal – which in this case implies that I is a complete intersection (see also Section 2). One condition that ensures that I is Cohen–Macaulay is the following. Assume that S $(=R/I)$ satisfies Serre's condition S_3 and that for each prime ideal $\mathbf{p}$ of S, depth $S_{\mathbf{p}} > \frac{1}{2}\dim S_{\mathbf{p}}$. As the canonical module of an almost complete intersection always satisfies the condition [1]

$$\operatorname{depth} W \geq \inf\{\dim S, 2+\operatorname{depth} S\},$$

again the duality theorem of [11] implies that I must be Cohen–Macaulay.

2. Koszul homology. I

Broadly speaking we shall view the homology of the Koszul complex $\mathbf{K}$ built on a set $\mathbf{x}$ of generators of the ideal I as an intermediary between the properties of the syzygies of I (over R) and those of I/I^2 (over R/I) (cf. [21]). Of particular concern shall be the depth properties of the modules $H_i = H_i(\mathbf{K})$ and the inherent duality.

A key constrution here is provided by Tate's resolution of R/I: There exists a graded algebra $\mathbf{T}=\bigoplus T_s$, T_s = finitely generated free R-module and a differential $d: T_s \to T_{s-1}$, which is a free resolution of R/I. It is obtained in the following manner. If $\mathbf{K}$ is acyclic, $\mathbf{T}=\mathbf{K}$. Otherwise $H_1(\mathbf{K})\neq 0$; add variables $U_1, \ldots, U_m$ of degree 2 to kill the generators of H_1 – i.e. construct a divided power algebra $\mathbf{X}=\mathbf{K}(U_1,\ldots,U_m)$, $H_1(\mathbf{X})=0$. This construction is carried over to higher cycles. Gulliksen enhanced this by showing how certain derivations (i.e. derivations commuting with the differential of $\mathbf{T}$) of degree -2 arise. Precisely, the last lines of the proof of [10, Theorem 1.4.9] may be slightly recast: Given a commutative diagram of R-maps

$$\begin{array}{ccc}
X_2 = K_2 \oplus RU_1 \oplus \cdots \oplus RU_m & & \\
\downarrow d & \searrow \partial & \\
Z_1(\mathbf{K}) & & R \\
\downarrow & & \downarrow \\
H_1(\mathbf{K}) & & \downarrow \\
& \searrow \sigma & \downarrow \\
& & R/I
\end{array}$$

where ∂ is trivial on K_2, it can be extended to a derivation of $\mathbf{T}$.

As in [10] this has the following consequences. If I is generated by $\{x_1, \ldots, x_n\}$, denote by N the ideal generated by the coefficients of the elements of Z_1 as a submodule of K_1, that is, N is the $(n-1)$-Fitting ideal of I. On the other hand, let L/I be the trace ideal of H_1 (over R/I), that is L/I = ideal generated by all $\sigma(e)$, $e \in H_1$, $\sigma \in H_1^*$.

(2.1) **Theorem.** *Let R be a Noetherian ring and let I be an ideal of finite projective dimension. Then $\sqrt{N} = \sqrt{L}$. In particular, if R is a local ring and $n = \upsilon(I)$, then $L \neq R$.*

(2.2) **Corollary.** *Let R be a local ring and let I be an ideal of finite projective dimension. Denote $n = \upsilon(I)$, $g = \mathrm{Ht}(I)$.*

(a) *If H_1 is (R/I-) free, then I is a complete intersection.*

(b) *If $\mathrm{pd}_{R/I}(H_1) \leq 1$, then $c(I) \leq n+1$.*

(c) *Let I be a Gorenstein ideal; if $\mathrm{pd}_{R/I}(H_{n-g-1}) \leq 1$, then $c(I) \leq \inf\{n+1, 2g+1\}$.*

Remark. It will follow from the discussion of Section 4 that condition (c) "I is a Gorenstein ideal" may be simply replaced by "R/I satisfies Serre's condition S_2". It turns out that $\mathrm{pd}_{R/I}(H_{n-g-1}) < \infty$ and the stated bound for $c(I)$ imply that the 'canonical' module, $W = H_{n-g}$, is actually R/I. This will follow from two observations:

(i) Since R/I satisfies S_2 and I is a complete intersection in codimension one, $R/I = \mathrm{Hom}_{R/I}(W, W)$.

(ii) There exists a natural mapping

$$\mathrm{Hom}_S(\mathrm{Hom}_S(E, W), W) \to \mathrm{Hom}_S(\mathrm{Hom}_S(E, S) \otimes W, W)$$
$$\xrightarrow{\sim} \mathrm{Hom}_S(\mathrm{Hom}_S(E, R), R).$$

One will then be allowed to identity H_{n-g} and the 'determinant' of H_{n-g-1}.

Proof of (2.2). (a) follows directly from (2.1). It was the original motivation of Gulliksen.

(b) Since I may, by (a), to be assumed to be generically a complete intersection, the rank of H_1 is $n-g$. Since it has projective dimension at most 1, by [5] – its free locus has codimension at most $n-g+1$. Seen in $\mathrm{Spec}(R)$ this is the asserted inequality.

Before proving (c) we shall need:

(2.3) **Lemma.** *Let $\{R, m\}$ be a local ring of depth at most 1 and let E be a finitely generated R-module. If $E^* = \mathrm{Hom}_R(E, R)$ is a free R-module, then the natural mapping $E \to E^{**}$ is a surjection.*

Proof. Denote by L the kernel of the natural mapping $E \to E^{**}$ and consider the induced exact sequence

$$0 \to L \to E \to E\hat{} \to 0.$$

Our assertion is that $E\hat{} = E^{**}$. Note that the mapping $E\hat{} \to (E\hat{})^{**}$ is an inclusion and $(E\hat{})^* = E^*$. Thus we may replace E by $E\hat{}$ – i.e. we may assume that $L = 0$.

Let I be the trace ideal of E; if $I = R$, E has a free summand and we can argue by induction on the free rank of E^*. Assume then $I \neq R$. If depth $R = 0$, this is a contradiction as otherwise I – and E^* along with it – would have a nontrivial annihilator. If depth $R = 1$, I^{-1} is strictly larger than R and $I^{-1} \cdot E^*$ is naturally embedded in E^*, a contradiction to the R-freeness of E^*. □

Proof of (2.2c). From the asserted bound for $c(I)$ we may assume that I is a complete intersection in codimension at most one. Recall that there is a natural pairing in $H(\mathbf{K})$

$$H_i(\mathbf{K}) \times H_{n-g-i}(\mathbf{K}) \to H_{n-g}(\mathbf{K})$$

which is an isomorphism whenever I is a complete intersection. This defines a mapping $H_1 \to (H_{n-g-1})^*$ that identifies the last module with the second dual, $(H_1)^{**}$, of H_1.

Consider the exact sequence

$$H_1 \to (R/I)^n \to I/I^2 \to 0.$$

Taking double duals, we get the exact sequence

$$0 \to H_1^{**} \to (R/I)^n \to (I/I^2)\hat{} \to 0,$$

where $(I/I^2)\hat{} = I/I^2$ modulo its torsion submodule. But the free locus of H_{n-g-1} – and therefore the free locus of its dual as well – has codimension at most $n-g+1$. But if $(H_1)^{**}$ is free, the free locus of $(I/I^2)\hat{}$ has codimension at most $n-(n-g)+1$. On the other hand, with $(I/I^2)\hat{}$ free of rank g it follows that I is a complete intersection (cf. [26]). □

3. The (co-)normal modules

In order to obtain yet sharper estimates for $c(I)$ we have placed constraints on the resolution of the ideal I – as done in Section 1 – or on the homology of the Koszul complex since such modules have a representative set of properties that are transmitted across its linkage class, as in Section 2. From here on we place restrictions directly on the conormal bundle I/I^2 itself. This will take the form of certain conjectures on the thus far observed homological rigidity of I/I^2 and of the other (co-)normal modules.

Let I be an ideal of the ring R (occasionally assumed to contain a field). In general R will be taken to be regular; at least we shall assume that I has finite projective dimension. If I has height g and R is a Gorenstein ring, let W be the canonical module of R/I, $\mathrm{Ext}_R^g(R/I, R)$.

We shall denote:
(1) conormal module (of the embedding $V(I)\subset \mathrm{Spec}(R)):=I/I^2$;
(2) normal module $:=(I/I^2)^*=\mathrm{Hom}(I/I^2, R/I)$;
(3) twisted conormal module $:=I/I^2\otimes W$.
In addition, if R is a polynomial ring over a field K of characteristic 0, we shall also consider the (co-)tangent modules:
(4) module of K-differentials $:=\Omega(R/I, K)$;
(5) module of K-derivations $:=\mathrm{Der}_K(R/I, R/I)$.
If E is any of these 5 modules:

Conjecture. If $\mathrm{pd}_{R/I}(E)<\infty$, then I is locally a complete intersection.

We focus on the conormal module I/I^2. The assertion is that for a regular local ring R and an ideal I, $\mathrm{pd}_{R/I}(I/I^2)=0$ or ∞.

Remarks. (a) An early instance of this conjecture is contained in the statement of Chevalley's theorem: If both R and $S=R/I$ are regular, then I is generated by a subset of a minimal generating set of the maximal ideal of R.

(b) A naive attempt at a counter-example: Let A be a regular local ring and let M be a finitely generated module. Put $R:=\mathrm{Sym}_A(M)=$ symmetric algebra of M. $I:=$ augmentation ideal of R. Then $I/I^2=M$ so that $\mathrm{pd}_{R/I}(I/I^2)<\infty$. But if A contains a field, using the homogeneous version of the zero-divisior conjecture, it is easy to see that $\mathrm{pd}_R(I)<\infty$ only occurs in case M is a free A-module.

4. Dimension one and the canonical module

The hypothesis that one of the R/I-modules E has finite projective dimension implies that I is unmixed. It is also easy to prove – or will become clear in the more general cases we shall discuss – that I is generically a complete intersection. Now we assemble the pieces for the determination of the canonical module of R/I.

In the remainder of the paper we shall keep the following notation: $S=R/I$, $n=\upsilon(I)$, $g=\mathrm{ht}(I)$ and $W=\mathrm{Ext}_R^g(R/I, R)$.

(4.1) **Proposition.** *If* $\mathrm{pd}_{R/I}(I/I^2$ *or* $I/I^2\otimes W)\leq 1$, *then* I *is a local complete intersection.*

Proof. For I/I^2 the assertion follows from (2.1).

We may assume that R is a local ring. For $E=I/I^2\otimes W$ we make a reduction to the previous case. Let

$$0\to L\to S^m\to W\to 0$$

and

$$0\to H\to S^n\to I/I^2\to 0$$

be minimal presentations of W and I/I^2. Tensoring the two presentations we get the exact sequence

$$L\otimes S^n\oplus S^m\otimes H\to S^m\otimes S^n\to W\otimes I/I^2\to 0.$$

Note that localizing at the associated primes of S we get, in both cases, that I/I^2 is free. Thus, by [7] or [23], I is a generic complete intersection. As a consequence the module $W\otimes I/I^2$ has rank g as well. This means that the term on the left must map onto a free module of rank $mn-g$. By (2.1) we know that H has no nontrivial free summand so that if r is the maximum rank of a free summand of L we must have $rn\geq mn-g$. Since $r\leq m-1$, $g\geq n$ and thus I/I^2 is a free R/I-module. By [7] or [23], I is then a complete intersection. □

Remarks. (a) For the normal module, $(I/I^2)^*$, the assertion above is not entirely known. Note, however, that if $\operatorname{depth}(R/I)\leq 1$, then it follows from (2.3) that I/I^2 has a free summand of rank g, and thus by [26] I must be a complete intersection.

(b) The module of differentials, $\Omega(R/I,K)$, could be added to the list of (4.1). In fact, $\operatorname{pd}_{R/I}\Omega(R/I,K)\leq 1$ is a characterization of local complete intersections (cf. [26]).

(c) The module of derivations, $\operatorname{Der}_K(R/I,R/I)$, being a dual module, is free in depth at most two. Taking into account the main result of [16], we would arrive at the same conclusion as (a).

We recall that there exists a canonical mapping (cf. [9, supplement])

$$\operatorname{Ext}_R^g(R/I,R)=W\to(\textstyle\bigwedge^g I/I^2)^*,$$

that is an isomorphism whenever I is a complete intersection in depth at most 1 (see also [20]). In the case of the modules of differentials or of derivations, we can also identify W to one of the following modules ($d=\dim R$):

$$(\textstyle\bigwedge^{d-g}\Omega(R/I,K))^{**},\qquad(\textstyle\bigwedge^{d-g}\operatorname{Der}_K(R/I,R/I))^*.$$

Therefore, if E is one of the modules in the conjecture – excepting for $I/I^2\otimes W$ – then $W=\det(E)$ or $W=\det(E)^*$. (Recall: $\det(E)=(\bigwedge^r E)^{**}$, where r is the generic rank of E. If E is a finitely generated S-module, of finite projective dimension, $\det(E)$ is an invertible ideal of S; see [17] for further details.)

(4.2) **Divisor Theorem.** *Let R be a regular local ring (or, appropriately, a localization of a polynomial ring over a field of characteristic 0) and let I be an ideal. Denote by $[W]$ the class of the canonical module of R/I in its the divisor class monoid. Then*

(a) *If* $\operatorname{pd}_{R/I}(I/I^2\otimes W)<\infty$, *then* $(g-1)[W]=0$.

(b) *If* $\operatorname{pd}_{R/I}(E=$ *one of the four other modules*$)<\infty$, *then* $[W]=0$.

Proof. The cases in (b) follow from the preceding discussion, while (a) is a consequence of (4.1) and the usual formula for the divisor associated to a tensor product. □

4.3. **Corollary.** *If $g=2$ and R contains a field, then I is a local complete intersection in all 5 cases.*

Proof. The previous result says that I is a height 2 unmixed ideal and $\mathrm{Ext}^2_R(R/I,R)=R/I$ (locally), so that we can appeal directly to [6]. □

5. Koszul homology. II

At its simplest the relationship between the syzygies of an ideal I and those of I/I^2 is already reflected in the exact sequences:

(i) $$0\to T_2^{S/R}\to H_1\to S^2\to I/I^2\to 0$$

and (under Cohen–Macaulay conditions and in the setting of the conjectures)

(ii) $$0\to (I/I^2)^*\to S^n\to H_{n-g-1}\to T^2_{S/R}\to 0,$$

where the T's denote the usual deformation functors (cf. [12]).

These sequences show already how intertwined the H_i's are to the syzygetic properties of I.

(5.1) **Proposition.** *Let I be an ideal of finite projective dimension. If H_1 is Cohen–Macaulay, then* $\mathrm{pd}_S(I/I^2)=0$ *or* ∞.

Proof. Since I is generically a complete intersection the hypothesis on H_1 implies that $T_2^{S/R}=0$. At this point (2.1) applies. □

There is an analogous result for $(I/I^2)^*$.

We point out now several instances where depth information on the Koszul homology is forthcoming. Assume, to simplify matters, that R is a regular local ring. In the sequel we may assume that I is a complete intersection on the punctured spectrum of R.

(a) Let I be an almost complete intersection, that is, $\upsilon(I)\leq g+1$. According to [1], depth $H_1\geq \inf\{\dim S, 2+\operatorname{depth} S\}>0$, so that if $\mathrm{pd}_S(I/I^2)<\infty$, from the sequence (i) we get that $T_2^{S/R}=0$ (since by assumption it has finite length) and H_1 has finite projective dimension as well. Therefore H_1 will be a module of rank one, satisfying Serre's S_2 condition and of finite projective dimension. It is thus free; again (2.1) applies.

(b) If I is Cohen–Macaulay and $\upsilon(I)\leq g+2$, it is shown in [2] that the Koszul homology is Cohen–Macaulay (see also [13]). What is not known is whether the conjecture holds true for all the cases with $\upsilon(I)=g+2$, e.g. for a non Cohen–Macaulay ideal of height 3 and 5 generators.

(c) A broad class of cases is taken care by a theorem of Huneke [14] who proved that the property "the Koszul homology is Cohen–Macaulay" is an invariant of the even linkage class of the ideal. It follows that if I is in the linkage class (not just even linkage class) of a complete intersection then its Koszul homology is Cohen–Macaulay. This extends the validity of the conjecture (for I/I^2) to Gorenstein ideals of height 3 and, therefore, by (4.1), to all Cohen–Macaulay ideals of height 3.

(d) A weaker version of (c) above is more widely observed: that depth $H_i \geq n-g+i$, a condition labelled 'sliding depth'. It is also an invariant of even linkage. Furthermore (cf. [13]): If I/I^2 is torsion-free, then H_1 is Cohen–Macaulay. This requires that R be a Gorenstein ring; it is now known whether the higher homology modules are also Cohen–Macaulay.

6. Low projective dimension

We now look at estimates for $c(I)$ for an ideal I where $\mathrm{pd}_{R/I}(I/I^2)$ is at most 3. We may assume that R is a local ring of dimension d and that I is a complete intersection on the punctured spectrum of R, that is, $d=c(I)$. Let

$$0 \to S^p \to S^q \to S^m \to S^n \to I/I^2 \to 0$$

be a minimal resolution of I/I^2 as an S $(=R/I)$-module.

(6.1) **Proposition.** *In the situation above assume further that* 2 *is invertible in* R. *We then have:*

(a) $n+1+p \geq d$;

(b) $m \geq \binom{n+1}{2} - g - p$.

Proof. (a) Because of (2.1a) we have only to determine the locus where the mapping $S^q \to S^m$ has a free summand of S^m for its image. Note that it has rank $t=q-p$. Taking into account that the sequence splits on the punctured spectrum of S we obtain, from (1.2), the following estimate

$$m+q-2(q-p)+1 \geq \dim(S) = d-g,$$

which is a rephrasing of (a).

(b) Consider a presentation of I

$$\begin{array}{c} 0 \to Z_2 \to R^m \to R^n \to I \to 0 \\ \searrow \;\; \nearrow \\ Z_1 \end{array}$$

Tensoring with S we get the exact sequences

$$0 \to \mathrm{Tor}_1^R(I,S) \to Z_1 \otimes S \to S^n \to I/I^2 \to 0,$$

$$0 \to \mathrm{Tor}_2^R(I,S) \to Z_2 \otimes S \to S^m \to Z_1 \otimes S \to 0.$$

By the hypothesis above the kernel L of the composite map

$$S^m \to Z_1 \otimes S \to \ker(S^n \to I/I^2)$$

has projective dimension at most one and rank $m-n-g=g-p$ – since I/I^2 has generic rank g.

On the other hand, since $1/2 \in R$, the anti-symmetrization map $\bigwedge^2 I \to I \otimes I$ splits and therefore $\bigwedge^2 I$ splits off the torsion submodule of $I \otimes I$ – that is, off $\mathrm{Tor}_1^R(I,S)$ – as well (see [21] for further details). Since, by the snake lemma, L maps onto $\mathrm{Tor}_1^R(I,S)$, its minimal number υ of generators must be at least $\binom{n}{2}$.

We thus have $\upsilon \leq m-n+g+p$, so that (b) follows. □

(6.2) **Corollary.** *Let I be a Cohen–Macaulay ideal of height* 4. *If* $\mathrm{pd}_S(I/I^2) \leq 3$ *and* $p<12$, *then I is a complete intersection.*

Proof. Since I must be a Gorenstein ideal (cf. 4.2), we have $m=2n-2$. We may assume that $n>6$ (cf. Section 5). Now use (b). □

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